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COMMENT

Comment on 'The collocation variational method for solving Fredholm integral equations . . .'

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Abstract. In a recent paper by Singh and Zerner, concerned with a numerical method for integral equations, a sufficient condition given for convergence is that certain points needed in the method be the roots of a 'reasonable' quadrature formula. It is pointed out that no quadrature formula can be 'reasonable' in the sense required, and that the theorem is false if the points are chosen in any other way.

In a recent paper, Singh and Zerner (1977) have discussed a particular numerical method for the solution of the integral equation

$$f(x)=g(x)+\lambda\int_a^b k(x,y)f(y)\,\mathrm{d}y, \tag{1}$$

or

$$f=g+\lambda Kf, \tag{2}$$

the equation being considered in the Hilbert space H of square integrable functions on $[a, b]$. In the present discussion we assume that a and b are finite, with $a<b$, and that the corresponding homogeneous equation $f=\lambda Kf$ has only the trivial solution.

The numerical method considered by Singh and Zerner employs an approximate solution of the form

$$f_n(x)=\sum_{j=1}^{n}\alpha_{nj}\psi_j, \tag{3}$$

where $\{\psi_j\}$ is a complete orthonormal set in H. The coefficients α_{nj} are determined by requiring f_n to satisfy the exact equation (1) at n selected points $x_{n1},\ldots,x_{nn}$, i.e.

$$f_n(x_{ni})=g(x_{ni})+\lambda\int_a^b k(x_{ni},y)f_n(y)\,\mathrm{d}y,\qquad i=1,\ldots,n. \tag{4}$$

This condition leads immediately to a set of n linear equations for the coefficients, namely

$$\sum_{j=1}^{n}\left(\psi_j(x_{ni})-\lambda\int_a^b k(x_{ni},y)\psi_j(y)\,\mathrm{d}y\right)\alpha_{nj}=g(x_{ni}),\qquad i=1,\ldots,n. \tag{5}$$

The method so described is referred to by Singh and Zerner as the collocation variational method. It is a reasonably attractive method in practice, provided that the set $\{\psi_j\}$ and the points $\{x_{nj}\}$ are carefully chosen, and has been discussed by many authors, including Phillips (1972) and Atkinson (1976), often under the name of the collocation method.

Singh and Zerner state a theorem to the effect that if K is compact, if (2) has a solution f in H, and if $x_{n1}, \ldots, x_{nn}$ are the roots of a 'reasonable' quadrature rule, then $\|f-f_n\| \to 0$ as $n \to \infty$, where the norm is the Hilbert space norm

$$\|f\| = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

A quadrature rule is 'reasonable', according to Singh and Zerner, if it has the property

$$\left| \int_a^b h(x) \, \mathrm{d}x - \sum_{k=1}^{n} w_{nk} h(x_{nk}) \right| \leqslant \frac{M}{n^{1+\epsilon}}, \tag{6}$$

for some $M < \infty$ and some $\epsilon > 0$. (Here w_{nk} is the weight associated with the root x_{nk}.) Though not stated explicitly, it appears from the proof that (6) is to hold (for the same value of M and ϵ) for every h in H.

However, *no* quadrature rule has the property (6) for all h in H, even if M and ϵ are allowed to depend on h. This assertion follows from a result of Lipow and Stenger (1972): if $\{\delta_n\}$ is any sequence of real numbers converging monotonically to zero, then there exists a continuous function h and an increasing sequence $\{n_i\}$ of positive integers such that

$$\int_a^b h(x) \, \mathrm{d}x - \sum_{k=1}^{n_i} w_{n_i k} h(x_{n_i k}) = \delta_i,$$

for all $i \geqslant 1$. For the present application we choose $\delta_n = 1/n$, and observe that the continuous function h so obtained belongs to H, but violates the condition (6) if $n = n_i$ and i is sufficiently large. (In a general way, the result of Lipow and Stenger asserts that given any quadrature rule, there exist continuous functions for which the convergence is arbitrarily slow.)

Although 'reasonable' quadrature formulae do not exist, one might wonder whether the theorem of Singh and Zerner could be retrieved by choosing the points in some other way. Unfortunately, the answer is no: for the theorem is false for *every* choice of the points $\{x_{ni}\}$. To prove that assertion, let the points $\{x_{ni}\}$ be chosen in any definite way, and consider the integral equation (1) with k identically zero, and with inhomogeneous term g given by

$$g(x) = \begin{cases} 0 & \text{if } x = x_{ni} \text{ for some } n \text{ and some } i,\ 1 \leqslant i \leqslant n, \\ 1 & \text{otherwise.} \end{cases}$$

Then the integral operator K is zero, and hence trivially compact, and the exact solution is $f = g$. Since f has the value 1 except on a set of measure zero, its norm is $\|f\| = (b-a)^{1/2}$. On the other hand, it follows from (5) and (3) that the approximate solution is $f_n = 0$. Thus $\|f_n - f\| = \|f\| = (b-a)^{1/2} \nrightarrow 0$, which contradicts the theorem.

In spite of the remarks in the preceding paragraph, there is no doubt that the method often works well in practice, particularly if g and f are continuous and have one or more continuous derivatives. Moreover, the convergence of $\|f_n - f\|$ to zero is often rigorously valid. Sufficient conditions for convergence are given, for example, by Phillips (1972) and Atkinson (1976), though not perhaps in a form that is very easy to use.

Briefly, the standard convergence result is as follows. Let P_n be the linear operator defined by

$$P_n h = \sum_{j=1}^{n} \beta_{nj} \psi_j,$$

where the coefficients β_{nj} are determined by the condition $P_n h(x_{ni}) = h(x_{ni})$, $i = 1, \ldots, n$, thus giving the equations

$$\sum_{j=1}^{n} \psi_j(x_{ni}) \beta_{nj} = h(x_{ni}), \qquad i = 1, \ldots, n.$$

Obviously P_n is well defined only if the $n \times n$ matrix $\psi_j(x_{ni})$ $(i, j = 1, \ldots, n)$ is non-singular, and we assume that this is the case. Then it can be shown, if both $\|g - P_n g\| \to 0$ and $\|K - P_n K\| \to 0$ as $n \to \infty$, that f_n exists for all n sufficiently large, and satisfies $\|f - f_n\| \to 0$.

(It should be remarked that the theoretical analysis is usually considered not in the Hilbert space H, but in the Banach space $C[a, b]$ of continuous functions with the norm

$$\|h\| = \max_{a \leqslant x \leqslant b} |h(x)|.$$

The main reason for this is that P_n is an unbounded operator in H, whereas in $C[a, b]$ it is bounded.)

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